Big q-Laguerre and q-Meixner polynomials and representations of the algebra $U_q(\mathrm{su}_{1,1})$

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Abstract

Diagonalization of a certain operator in irreducible representations of the positive discrete series of the quantum algebra $U_q(\mathrm{su}_{1,1})$ is studied. Spectrum and eigenfunctions of this operator are found in an explicit form. These eigenfunctions, when normalized, constitute an orthonormal basis in the representation space. The initial $U_q(\mathrm{su}_{1,1})$-basis and the basis of eigenfunctions are interrelated by a matrix with entries, expressed in terms of big q-Laguerre polynomials. The unitarity of this connection matrix leads to an orthogonal system of functions, which are dual with respect to big q-Laguerre polynomials. This system of functions consists of two separate sets of functions, which can be expressed in terms of q-Meixner polynomials $M_n(x; b, c; q)$ either with positive or negative values of the parameter b. The orthogonality property of these two sets of functions follows directly from the unitarity of the connection matrix. As a consequence, one obtains an orthogonality relation for the q-Meixner polynomials $M_n(x; b, c; q)$ with $b < 0$. A biorthogonal system of functions (with respect to the scalar product in the representation space) is also derived.

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1. Introduction

The significance of representations of Lie groups and Lie algebras for studying orthogonal polynomials and special functions is well known. The appearance of quantum groups and quantized universal enveloping algebras (quantum algebras) and development of their representations led to their applications for elucidating properties of q-orthogonal polynomials and q-special functions (see, for example, [1–3]). Since the theory of quantum groups and their representations is much more complicated than the Lie theory, the corresponding applications in the theory of orthogonal polynomials and special functions are more difficult. At the first stage of such applications, the compact quantum groups and their finite dimensional representations have been used.

It is known that the noncompact Lie group $SU(1,1) \sim SL(2,\mathbb{R})$ and its representations are very productive for the theory of orthogonal polynomials and special functions. Unfortunately, there are difficulties with a satisfactory definition of the noncompact quantum group (or algebras of functions on the quantum group) $SU_q(1,1)$, which would give us a possibility to use such quantum group extensively for deeper understanding the theory of orthogonal polynomials and special functions. For this reason, representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$ have been commonly used for such purposes (see, for example, [4–7]).

In this paper we use representations of the positive discrete series of the quantum algebra $U_q(\mathrm{su}_{1,1})$ for exploring properties of big q-Laguerre polynomials and q-Meixner polynomials. In fact, we deal with certain operators in these representations and do not touch the Hopf

structure of the algebra $U_q(\mathrm{su}_{1,1})$. Our study of these polynomials is related to representation operators, which can be represented by a Jacobi matrix. We deal with those properties of big q-Laguerre polynomials and q-Meixner polynomials, which have not been usually investigated by means of group theoretical methods. Namely, we consider a representation operator A, represented by some particular Jacobi matrix. We diagonalize this selfadjoint bounded operator with the aid of big q-Laguerre polynomials. The orthogonality relation for these polynomials allows us to find a spectrum of the operator A. It is simple and discrete. We find an explicit form of all eigenfunctions of this operator. Since the spectrum is simple, these eigenfunctions constitute an orthogonal basis in the representation space. Then we normalize this basis. As a result, one has two orthonormal bases in the representation space: the canonical (or the initial) basis and the basis of eigenfunctions of the operator A. They are interrelated by a unitary matrix U with entries u_{mn}, which are explicitly expressed in terms of big q-Laguerre polynomials. Since $U^* U = U U^* = I$, there are two requirements for providing the unitarity of this matrix U, namely:

$$\sum_{n=0}^{\infty} u_{mn} u_{m'n} = \delta_{mm'}, \quad \sum_{m=0}^{\infty} u_{mn} u_{mn'} = \delta_{nn'}. \tag{1.1}$$

The first relation expresses the orthogonality relation for big q-Laguerre polynomials. In order to interpret the second relation, we consider big q-Laguerre polynomials $P_n(q^{-m}; a, b; q)$ as functions of n. In this way one obtains a set of orthogonal functions, which are expressed in terms of two sets of q-Meixner polynomials (these two sets of q-Meixner polynomials can be considered as a dual set of polynomials with respect to big q-Laguerre polynomials: such duality property is well known in the case of discrete polynomials, orthogonal on a finite set of points). Consequently, the second relation in (1.1) leads to the orthogonality relations for those q-Meixner polynomials, which enter to our two sets (with respect to certain measures). For one of these sets we obtain the well-known orthogonality relation for q-Meixner polynomials. The second set leads to an orthogonality relation for q-Meixner polynomials $M_n(x; b, c; q)$ with $b < 0$. As far as we know, the possibility of extending the common orthogonality relation for q-Meixner polynomials $M_n(x; b, c; q)$ to a wider range of the parameter b has not been discussed in the literature.

We deduce from orthogonality relations for functions, dual to the big q-Laguerre polynomials, that q-Meixner polynomials are associated with the indeterminate moment problem and the commonly known orthogonality measure for these polynomials is not an extremal one. (Note that if this measure would be extremal, then the set of the q-Meixner polynomials would form a basis in the corresponding Hilbert space.)

We consider also the case of two representation operators A_1 and A_2, such that $A_1^* = A_2$. Diagonalization of these operators leads to two sets of functions, which are biorthogonal with respect to the scalar product in the representation space.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We use the theory of special functions and notations of the standard q-analysis (see, for example, [8] and [9]). We use q-numbers $[a]_q$ defined as

$$[a]_q = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}, \tag{1.2}$$

where a is any complex number. We shall also use the well known notations

$$(a; q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad (a_1, \cdots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n.$$

2. Discrete series representations of $U_q(\mathrm{su}_{1,1})$

The quantum algebra $U_q(\mathrm{su}_{1,1})$ is defined as the associative algebra, generated by the elements J_+, J_-, and J_0, satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}} \equiv [2J_0]_q,$$

and the conjugation relations

$$J_0^* = J_0, \qquad J_+^* = J_-. \tag{2.1}$$

(Observe that here we have replaced J_- by $-J_-$ in the usual definition of the algebra $U_q(\mathrm{sl}_2)$.)

We are interested in the discrete series representations of $U_q(\mathrm{su}_{1,1})$ with lowest weights. These irreducible representations will be denoted by T_l^+, where l is a lowest weight, which can take any positive number (see, for example, [10]).

The representation T_l^+ can be realized on the space $\mathcal{L}_l$ of all polynomials in x. We choose a basis for this space, consisting of the monomials

$$f_n^l \equiv f_n^l(x) := c_n^l\, x^n, \quad n = 0, 1, 2, \cdots, \tag{2.2}$$

where

$$c_0^l = 1, \qquad c_n^l = \prod_{k=1}^{n} \frac{[2l+k-1]_q^{1/2}}{[k]_q^{1/2}} = q^{(1-2l)n/4} \frac{(q^{2l};q)_n^{1/2}}{(q;q)_n^{1/2}}\,, \quad n = 1, 2, 3, \cdots. \tag{2.3}$$

The representation T_l^+ is then realized by the operators

$$J_0 = x\frac{d}{dx} + l, \qquad J_\pm = x^{\pm 1}[J_0(x) \pm l]_q\,.$$

As a result of this realization, we have

$$J_+\, f_n^l = \sqrt{[2l+n]_q\,[n+1]_q}\, f_{n+1}^l = \frac{q^{-(n+l-1/2)/2}}{1-q}\sqrt{(1-q^{n+1})(1-q^{2l+n})}\, f_{n+1}^l, \tag{2.4}$$

$$J_-\, f_n^l = \sqrt{[2l+n-1]_q\,[n]_q}\, f_{n-1}^l = \frac{q^{-(n+l-3/2)/2}}{1-q}\sqrt{(1-q^{n})(1-q^{2l+n-1})}\, f_{n-1}^l, \tag{2.5}$$

$$J_0\, f_n^l = (l+n)\, f_n^l. \tag{2.6}$$

We know that the discrete series representations T_l can be realized on a Hilbert space, on which the conjugation relations (2.1) are satisfied. In order to obtain such a Hilbert space, we assume that the monomials $f_n^l(x)$, $n = 0, 1, 2, \cdots$, constitute an orthonormal basis for this Hilbert space. This introduces a scalar product $\langle\cdot,\cdot\rangle$ into the space $\mathcal{L}_l$. Then we close this space with respect to this scalar product and obtain the Hilbert space, which will be denoted by $\mathcal{H}_l$.

3. Representation operators related to big q-Laguerre polynomials

In this section we are interested in the operator

$$A := \alpha q^{J_0/4}\left(\sqrt{1-bq^{J_0-l}}\,J_+\,q^{(J_0-l)/2} + q^{(J_0-l)/2}J_-\sqrt{1-bq^{J_0-l}}\right) q^{J_0/4} - \beta_1 q^{2J_0} + \beta_2 q^{J_0-l} \tag{3.1}$$

of the representation T_l^+, where $b<0$ and

$$\alpha=(-b)^{1/2}q^l(1-q), \quad \beta_1=b(1+q), \quad \beta_2=bq+q^{2l}(b+1).$$

Since the bounded operator q^{J_0} is diagonal in the basis f_n^l, $n=0,1,2,\cdots$, without zero diagonal elements, the operator A is well defined.

We have the following expression for the symmetric operator A in the canonical basis f_n^l, $n=0,1,2,\cdots$:

$$A\,f_n^l=(-ab)^{1/2}q^{(n+2)/2}\Big[\sqrt{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})}\,f_{n+1}^l$$
$$+\,q^{-1/2}\sqrt{(1-q^n)(1-aq^n)(1-bq^n)}\,f_{n-1}^l\Big]$$
$$-[abq^{2n+1}(1+q)-q^{n+1}(a+ab+b)]f_n^l, \qquad a=q^{2l-1}.$$

Since $q<1$ the operator A is bounded. Therefore, one can close this operator and we assume in what follows that A is a closed (and consequently defined on the whole space $\mathcal{H}_l$) operator. Since A is symmetric, its closure is a selfadjoint operator.

We wish to find eigenfunctions $\xi_\lambda(x)$ of the operator A, $A\xi_\lambda(x)=\lambda\xi_\lambda(x)$. We set

$$\xi_\lambda(x)=\sum_{n=0}^{\infty}a_n(\lambda)f_n^l(x).$$

Acting by the operator A upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty}a_n(\lambda)\Big\{q^{(n+2)/2}(-ab)^{1/2}\sqrt{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})}\,f_{n+1}^l$$
$$+\,q^{(n+1)/2}(-ab)^{1/2}\sqrt{(1-q^n)(1-aq^n)(1-bq^n)}\,f_{n-1}^l+d_nf_n^l\Big\}=\lambda\sum_{n=0}^{\infty}a_n(\lambda)f_n^l,$$

where

$$d_n=-abq^{2n+1}(1+q)+q^{n+1}(a+ab+b).$$

Note that $0<a<q^{-1}$ since l can be any positive number. Comparing coefficients of a fixed f_n^l, one obtains a three-term recurrence relation for the coefficients $a_n(\lambda)$:

$$q^{(n+2)/2}(-ab)^{1/2}\sqrt{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})}\,a_{n+1}(\lambda)$$
$$+\,q^{(n+1)/2}(-ab)^{1/2}\sqrt{(1-q^n)(1-aq^n)(1-bq^n)}\,a_{n-1}(\lambda)+d_na_n(\lambda)=\lambda a_n(\lambda).$$

We make here the substitution

$$a_n(\lambda)=(-ab)^{-n/2}q^{-n(n+3)/4}\left(\frac{(aq,bq;q)_n}{(q;q)_n}\right)^{1/2}a_n'(\lambda)$$

and derive the relation

$$(1-aq^{n+1})(1-bq^{n+1})a_{n+1}'(\lambda)-abq^{n+1}(1-q^n)a_{n-1}'(\lambda)+d_na_n'(\lambda)=\lambda a_n'(\lambda),$$

where $0 < a < q^{-1}$ and $b < 0$. It coincides with the recurrence relation for the big q-Laguerre polynomials

$$\begin{aligned} P_n(\lambda; a, b; q) &:= {}_3\phi_2(q^{-n}, 0, \lambda;\ aq, bq;\ q, q) \\ &= (q^{-n}/b; q)_n^{-1}\ {}_2\phi_1(q^{-n},\ aq/\lambda;\ aq;\ q, \lambda/b) \end{aligned} \tag{3.2}$$

(see formula (3.11.3) in [11]), that is, $a'_n(\lambda) = P_n(\lambda; a, b; q)$, $a = q^{2l-1}$. Therefore,

$$a_n(\lambda) = (-ab)^{-n/2} q^{-n(n+3)/4} \left(\frac{(aq, bq; q)_n}{(q; q)_n} \right)^{1/2} P_n(\lambda; a, b; q). \tag{3.3}$$

Thus, the eigenfunctions of the operator A have the form

$$\xi_\lambda(x) = \sum_{n=0}^{\infty} (-ab)^{-n/2} q^{-n(n+3)/4} \left(\frac{(aq, bq; q)_n}{(q; q)_n} \right)^{1/2} P_n(\lambda; a, b; q) f_n^l(x) \tag{3.4}$$

$$= \sum_{n=0}^{\infty} (-b)^{-n/2} a^{-3n/4} q^{-n(n+3)/4} \frac{(aq; q)_n}{(q; q)_n} (bq; q)_n^{1/2} P_n(\lambda; a, b; q) x^n.$$

To find a spectrum of the operator A, we take into account the following. The selfadjoint operator A is represented by a Jacobi matrix in the basis $f_n^l(x)$, $n = 0, 1, 2, \cdots$. According to the theory of operators of such type (see, for example, [12], Chapter VII), eigenfunctions ξ_λ of this operator are expanded into series in the monomials $f_n^l(x)$, $n = 0, 1, 2, \cdots$, with coefficients, which are polynomials in λ. These polynomials are orthogonal with respect to some measure $d\mu(\lambda)$ (moreover, for selfadjoint operators this measure is unique). The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator under consideration and the spectrum is simple. Let us apply these assertions to the operator A.

The orthogonality relation for the big q-Laguerre polynomials $P_n(\lambda; a, b; q)$ for $0 < a < q^{-1}$ and $b < 0$ is known to be of the form

$$\begin{aligned} &\int_{bq}^{aq} \frac{(x/a, x/b; q)_\infty}{(x; q)_\infty} P_m(x; a, b; q) P_{m'}(x; a, b; q) d_q x \\ &\equiv \sum_{n=0}^{\infty} \frac{(q^{n+1}; q)_\infty (aq^{n+1}/b; q)_\infty}{(aq^{n+1}; q)_\infty} q^n P_m(aq^{n+1}; a, b; q) P_{m'}(aq^{n+1}; a, b; q) \\ &- \frac{b}{a} \sum_{n=0}^{\infty} \frac{(q^{n+1}; q)_\infty (bq^{n+1}/a; q)_\infty}{(bq^{n+1}; q)_\infty} q^n P_m(bq^{n+1}; a, b; q) P_{m'}(bq^{n+1}; a, b; q) \\ &= \frac{(q, b/a, aq/b; q)_\infty}{(aq, bq; q)_\infty} \frac{(q; q)_m}{(aq, bq; q)_m} (-ab)^m q^{m(m+3)/2} \delta_{mm'}. \end{aligned} \tag{3.5}$$

Notice that for $m = m' = 0$ the orthogonality relation (3.5) reduces to

$$\sum_{n=0}^{\infty} \frac{(q^{n+1}, aq^{n+1}/b; q)_\infty}{(aq^{n+1}; q)_\infty} q^n - \frac{b}{a} \sum_{n=0}^{\infty} \frac{(q^{n+1}, bq^{n+1}/a; q)_\infty}{(bq^{n+1}; q)_\infty} q^n = \frac{(q, aq/b, b/a; q)_\infty}{(aq, bq; q)_\infty}. \tag{3.6}$$

In terms of the ${}_2\phi_1$ basic hypergeometric series this identity can be written as

$$\frac{(aq/b, q; q)_\infty}{(aq; q)_\infty}\ {}_2\phi_1(aq, 0;\ aq/b;\ q, q)$$

$$
-\frac{b}{a}\frac{(bq/a,q;q)_\infty}{(bq;q)_\infty}\,{}_2\phi_1(bq,0;\ bq/a;\ q,q)=\frac{(aq/b,b/a,,q;q)_\infty}{(aq,bq;q)_\infty} \tag{3.7}
$$

and it represents a particular case of Sears' three-term transformation formula for ${}_2\phi_1$ series (see [8], formula (3.3.5)). So the orthogonality relation (3.5) is based on the relation (3.7). A detailed derivation of this transformation formula and the orthogonality relation (3.5) can be found in [13].

The explicit form of the orthogonality relation (3.5) is important for the case under discussion because it directly leads to the following statement.

Theorem 1. *The spectrum of the operator A coincides with the set of points aq^{n+1}, bq^{n+1}, $n=0,1,2,\cdots$. The spectrum is simple and it has only one accumulation point at 0.*

4. Representations T_l^+, related to big q-Laguerre polynomials

The operator A has eigenfunctions $\xi_\lambda(x)$, corresponding to the eigenvalues aq^{n+1}, bq^{n+1}, $n=0,1,2,\cdots$. Since these eigenvalues are distinct, the set of functions

$$
\Xi_n(x)\equiv\xi_{aq^{n+1}}(x),\quad \Xi'_n(x)\equiv\xi_{bq^{n+1}}(x)\qquad n=0,1,2,\cdots,
$$

constitutes an orthogonal basis in the representation space $\mathcal{H}_l$. For convenience, we often denote this basis as

$$
\tilde{\Xi}_n(x)\qquad n=0,\pm1,\pm2,\cdots,
$$

where $\tilde{\Xi}_n(x):=\Xi'_n(x)$, $n=0,1,2,\cdots$, and $\tilde{\Xi}_{-m}(x):=\Xi'_{m-1}(x)$, $m=1,2,3,\cdots$. This basis is not orthonormal. Let us find the orthonormal basis

$$
\hat{\Xi}_n(x)=c_n\tilde{\Xi}_n(x),\qquad n=0,\pm1,\pm2,\cdots, \tag{4.1}
$$

where c_n are normalization constants.

Observe that due to (3.4) the bases f_n^l, $n=0,1,2,\cdots$, and $\hat{\Xi}_n(x)$, $n=0,\pm1,\pm2,\cdots$, are connected by the formulas

$$
\hat{\Xi}_n(x)=c_n\sum_{m=0}^\infty a_m(aq^{n+1})f_m^l(x),\quad \hat{\Xi}'_n(x)=c'_n\sum_{m=0}^\infty a_m(bq^{n+1})f_m^l(x),\quad n=0,1,2,\cdots,
$$

where $a_m(\lambda)$ are defined in (3.3) and $\hat{\Xi}'_{n-1}(x)=\hat{\Xi}_{-n}(x)$ for $c'_{n-1}=c_{-n}$, $n=1,2,\cdots$. In order to find the coefficients c_n we take into account the following. The basis $\hat{\Xi}_n(x)$, $n=0,\pm1,\pm2,\cdots$, is orthonormal if the matrix (u_{mn}) with entries $u_{mn}=c_na_m(\lambda_n)$, where $\lambda_n=aq^{n+1}$ for $n=0,1,2,\cdots$ and $\lambda_n=bq^{-n}$ for $n=-1,-2,\cdots$, is unitary, that is, $\sum_n u_{mn}u_{m'n}=\delta_{mm'}$. Taking into account the explicit form (3.3) of the coefficients $a_m(\lambda_n)$ and the orthogonality relation (3.5) for the big q-Laguerre polynomials, we find that

$$
c_n=\left(\frac{(q^{n+1},aq^{n+1}/b,aq,bq;q)_\infty\,q^n}{(aq^{n+1},q,b/a,aq/b;q)_\infty}\right)^{\frac12}=\left(\frac{(aq;q)_n(bq;q)_\infty\,q^n}{(aq/b,q;q)_n(b/a;q)_\infty}\right)^{\frac12} \tag{4.2}
$$

and

$$
c'_n=\left(\frac{(-b/a)q^n(bq^{n+1}/a,q^{n+1},aq,bq;q)_\infty}{(bq^{n+1},q,b/a,aq/b;q)_\infty}\right)^{\frac12}=\left(\frac{(-b/a)q^n(bq;q)_n(aq;q)_\infty}{(q;q)_n(aq/b;q)_\infty(b/a;q)_{n+1}}\right)^{\frac12}. \tag{4.3}
$$

Namely, at these values of c_n and c'_n the formula $\sum_n u_{mn}u_{m'n}=\delta_{mm'}$ is equivalent to the orthogonality relation for the big q-Laguerre polynomials.

We know that the operator A acts upon the basis $\Xi_n(x)$, $\Xi'_n(x)$, $n = 0,1,2,\cdots$, in the following way:

$$A\,\Xi_n(x) = aq^{n+1}\Xi_n(x), \quad A\,\Xi'_n(x) = bq^{n+1}\Xi'_n(x),$$

Let us find how the operator q^{-J_0} acts upon this basis. From the q-difference equation (3.11.5) in [11], it follows that

$$q^{-n}\lambda^2 P_n(\lambda) = B(\lambda)P_n(q\lambda) - [B(\lambda)+D(\lambda)-\lambda^2]P_n(\lambda) + D(\lambda)P_n(q^{-1}\lambda),$$

where $P_n(\lambda) := P_n(\lambda;a,b;q)$, $B(\lambda) = abq(1-\lambda)$, and $D(\lambda) = (\lambda-aq)(\lambda-bq)$. We multiply both sides of this relation by $b_n f_n^l(x)$, where b_n is the constant factor in front of the $P_n(\lambda;a,b;q)$ on the right-hand side of (3.3), and sum up over n. Taking into account formula (3.4) and the relation $q^{-J_0}f_n^l = q^{-l-n}f_n^l$, we obtain

$$q^{-J_0+l}\lambda^2\xi_\lambda(x) = B(\lambda)\xi_{q\lambda}(x) - [B(\lambda)+D(\lambda)-\lambda^2]\xi_\lambda(x) + D(\lambda)\xi_{q^{-1}\lambda}(x).$$

Since $B(\lambda)+D(\lambda)-\lambda^2 = abq(1+q) - \lambda q(ab+a+b)$, we have

$$\begin{aligned} q^{-J_0}\Xi_n = a^{-3/2}bq^{-2n-3/2}(1-aq^{n+1})\Xi_{n+1} - a^{-3/2}q^{-2n-3/2}[b(1+q)-q^{n+1}(ab+a+b)]\Xi_n \\ + a^{-3/2}bq^{-2n-1/2}(1-q^n)(1-aq^n/b)\Xi_{n-1} \end{aligned} \tag{4.4}$$

for $\lambda = aq^{n+1}$, that is, for $\xi_{aq^{n+1}}(x) = \Xi_n(x)$, and

$$\begin{aligned} q^{-J_0}\Xi'_n = a^{1/2}b^{-1}q^{-2n-3/2}(1-bq^{n+1})\Xi'_{n+1} - a^{1/2}b^{-1}q^{-2n-3/2} \\ \times[1+q-a^{-1}q^{n+1}(ab+a+b)]\Xi'_n + a^{1/2}b^{-1}q^{-2n-1/2}(1-q^n)(1-bq^n/a)\Xi'_{n-1} \end{aligned} \tag{4.5}$$

for $\lambda = bq^{n+1}$, that is, for $\xi_{bq^{n+1}}(x) = \Xi'_n(x)$. Passing in (4.4) and (4.5) to the orthonormal basis $\hat{\Xi}_n$, $n = 0,\pm1,\pm2,\cdots$, we obtain

$$\begin{aligned} q^{-J_0}\hat{\Xi}_n = a^{-3/2}bq^{-2n-2}\sqrt{(1-aq^{n+1})(1-q^{n+1})(1-aq^{n+1}/b)}\,\hat{\Xi}_{n+1} \\ -a^{-3/2}q^{-2n-3/2}[b(1+q)-q^{n+1}(ab+a+b)]\,\hat{\Xi}_n \\ +a^{-3/2}bq^{-2n}\sqrt{(1-aq^n)(1-q^n)(1-aq^n/b)}\,\hat{\Xi}_{n-1} \end{aligned}$$

for $\hat{\Xi}_n$, $n = 0,1,2,\cdots$, and

$$\begin{aligned} q^{-J_0}\hat{\Xi}'_n = a^{1/2}b^{-1}q^{-2n-2}\sqrt{(1-bq^{n+1})(1-q^{n+1})(1-bq^{n+1}/a)}\,\hat{\Xi}'_{n+1} \\ -a^{1/2}b^{-1}q^{-2n-3/2}[1+q-a^{-1}q^{n+1}(ab+a+b)]\,\Xi'_n \\ +a^{1/2}b^{-1}q^{-2n}\sqrt{(1-bq^n)(1-q^n)(1-bq^n/a)}\,\hat{\Xi}'_{n-1}. \end{aligned}$$

The operators A and q^{-J_0} completely determine the representation T_l^+ in the basis $\hat{\Xi}_n$, $n = 0,\pm1,\pm2,\cdots$ (see, for example, [14]). However, the expressions for the operators J_+ and J_- are rather complicated.

5. Dual polynomials and functions

The matrix (u_{mn}), $m = 0,1,2,\cdots$, $n = 0,\pm1,\pm2,\cdots$, with entries $u_{mn} = c_n a_m(\lambda_n)$, described in the previous section, is unitary and it connects two orthonormal bases in the Hilbert space $\mathcal{H}_l$. The unitarity of this matrix means that the following relations hold:

$$\sum_{n\in\mathbb{Z}} u_{mn}u_{m'n} = \delta_{mm'}, \quad \sum_{m=0}^{\infty} u_{mn}u_{mn'} = \delta_{nn'}. \tag{5.1}$$

It is easy to see that the first relation is equivalent to the orthogonality relation for the big q-Laguerre polynomials (see the previous section). The second relation is the orthogonality relation for the functions, which are dual to the big q-Laguerre polynomials, and are defined as

$$f_n(q^{-m}; a, b|q) := P_m(aq^{n+1}; a, b; q), \quad n = 0, 1, 2, \cdots, \tag{5.2}$$

$$g_n(q^{-m}; a, b|q) := P_m(bq^{n+1}; a, b; q), \quad n = 0, 1, 2, \cdots. \tag{5.3}$$

Taking into account the expressions for the entries u_{mn}, the relation $\sum_{m=0}^{\infty} u_{mn}u_{mn'} = \delta_{nn'}$ can be written as

$$\sum_{m=0}^{\infty} a_m(\lambda_n)a_m(\lambda_{n'}) = c_n^{-2}\delta_{nn'},$$

where c_n must be replaced by c'_n if $\lambda_n = bq^{n+1}$. Substituting the explicit expressions for the coefficients $a_m(\lambda_n)$, we derive the following orthogonality relations for the functions (5.2) and (5.3):

$$\sum_{m=0}^{\infty} \frac{(aq, bq; q)_m}{(q; q)_m(-abq^2)^m} q^{-m(m-1)/2} f_n(q^{-m}; a, b|q) f_{n'}(q^{-m}; a, b|q) = c_n^{-2}\delta_{nn'}, \tag{5.4}$$

$$\sum_{m=0}^{\infty} \frac{(aq, bq; q)_m}{(q; q)_m(-abq^2)^m} q^{-m(m-1)/2}\, g_n(q^{-m}; a, b|q)\, g_{n'}(q^{-m}; a, b|q) = {c'}_n^{-2}\delta_{nn'}, \tag{5.5}$$

$$\sum_{m=0}^{\infty} \frac{(aq, bq; q)_m}{(q; q)_m(-abq^2)^m} q^{-m(m-1)/2}\, f_n(q^{-m}; a, b|q)\, g_{n'}(q^{-m}; a, b|q) = 0, \tag{5.6}$$

where c_n and c'_n are given by the formulas (4.2) and (4.3).

Comparing the expression (3.13.1) in [11] for the q-Meixner polynomials

$$M_n(q^{-x}; a, b; q) := {}_2\phi_1(q^{-n}, q^{-x};\ aq;\ q, -q^{n+1}/b)$$

with the explicit form (3.2) of the big q-Laguerre polynomials $P_m(x; a, b; q)$, we see that

$$f_n(q^{-m}; a, b|q) = (q^{-m}/b; q)_m^{-1} M_n(q^{-m}; a, -b/a; q).$$

Since $(q^{-m}/b; q)_m = (bq; q)_m(-b)^{-m}q^{-m(m+1)/2}$, the orthogonality relation (5.4) leads to the orthogonality relation for the q-Meixner polynomials $M_n(q^{-m}) \equiv M_n(q^{-m}; a, -b/a; q)$:

$$\sum_{m=0}^{\infty} \frac{(aq; q)_m(-b/a)^m q^{m(m-1)/2}}{(bq, q; q)_m} M_n(q^{-m})M_{n'}(q^{-m})$$

$$= \frac{(b/a; q)_\infty}{(bq; q)_\infty} \frac{(aq/b, q; q)_n}{(aq; q)_n} q^{-n}\delta_{nn'}, \tag{5.7}$$

where, as before, $0 < a < q^{-1}$ and $b < 0$. This orthogonality relation coincides with formula (3.13.2) in [11].

The functions (5.3) are also expressed in terms of q-Meixner polynomials. Indeed, we have

$$g_n(q^{-m}; a, b|q) = {}_3\phi_2(q^{-m}, 0, bq^{n+1};\ aq, bq;\ q, q)$$

$$= (q^{-m}/a; q)_m^{-1}\, {}_2\phi_1(q^{-n}, q^{-m};\ bq;\ q, bq^{n+1}/a)$$

$$= (q^{-m}/a; q)_m^{-1}\, M_n(q^{-m}; b, -a/b;\ q),$$

where $b<0$, that is, one of the parameters in these q-Meixner polynomials is negative.

Substituting this expression for $g_n(q^{-m};a,b|q)$ into (5.5), we obtain the orthogonality relation for q-Meixner polynomials $M_n(q^{-m})\equiv M_n(q^{-m};b,-a/b;\ q)$ with negative b:

$$\sum_{m=0}^{\infty}\frac{(bq;q)_m(-a/b)^m}{(aq,q;q)_m}\,q^{m(m-1)/2}\,M_n(q^{-m})M_{n'}(q^{-m})$$

$$=\frac{(a/b;q)_\infty}{(aq;q)_\infty}\frac{(bq/a,q;q)_n}{(bq;q)_n}q^{-n}\delta_{nn'}. \tag{5.8}$$

Observe that this orthogonality relation is of the same form as for $b>0$ (see, for example, formula (3.13.2) in [11]). As far as we know, this type of orthogonality relation for negative values of the parameter b has not been discussed in the literature.

The relation (5.6) can be written as the equality

$$\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m-1)/2}}{(q;q)_m}M_n(q^{-m};a,-b/a;\ q)M_{n'}(q^{-m};b,-a/b;\ q)=0,$$

which holds for $n,n'=0,1,2,\cdots$. The reader is invited to verify directly the validity of this identity for arbitrary nonnegative integers n and n' by using Jackson's q-exponential function

$$E_q(z):=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}z^n=(-z;q)_\infty$$

and the fact that $E_q(z)$ has zeroes at the points $z_j=-q^{-j}$, $j=0,1,2,\cdots$, namely, $E_q(-q^{-j})=0$.

Notice that the appearance of the q-Meixner polynomials here as a dual family with respect to the big q-Laguerre polynomials is quite natural because the transformation $q\to q^{-1}$ interrelates these two sets of polynomials, that is,

$$M_n(x;b,c;q^{-1})=(q^{-n}/b;q)_n\,P_n(qx/b;1/b,-c;q).$$

Let us introduce the Hilbert space $\mathfrak{l}^2_b$ of functions $F(q^{-m})$ on the set $m\in\{0,1,2,\cdots\}$ with a scalar product given by the formula

$$\langle F_1,F_2\rangle_b=\sum_{m=0}^{\infty}\frac{(aq,bq;q)_m}{(q;q)_m(-abq^2)^m}\,q^{-m(m-1)/2}\,F_1(q^{-m})\overline{F_2(q^{-m})}. \tag{5.9}$$

Now we can formulate the following statement.

Theorem 2. *The functions (5.2) and (5.3) constitute an orthogonal basis in the Hilbert space* $\mathfrak{l}^2_b$.

Proof. To show that the system of functions (5.2) and (5.3) constitutes a complete basis in the space $\mathfrak{l}^2_b$ we take in $\mathfrak{l}^2_b$ the set of functions F_k, $k=0,1,2,\cdots$, such that $F_k(q^{-m})=\delta_{km}$. It is clear that these functions constitute a basis in the space $\mathfrak{l}^2_b$. Let us show that each of these functions F_k belongs to the closure $\bar{V}$ of the linear span V of the functions (5.2) and (5.3). This will prove the theorem. We consider the functions

$$\hat{F}_k(q^{-m})=\sum_{n=-\infty}^{\infty}u_{kn}u_{mn},\qquad k=0,1,2,\cdots,$$

where u_{jn} are the same as in (5.1). Then $\hat{F}_k(q^{-m}) \in \bar{V}$ and, due to the first equality in (5.1), $\hat{F}_k$, $k = 0, 1, 2, \cdots$, coincide with the corresponding functions F_k, introduced above. The theorem is proved.

The measure in (5.9) does not coincide with the orthogonality measure for q-Meixner polynomials. Multiplying the measure in (5.9) by $[(bq;q)_m(-b)^{-m}q^{-m(m+1)/2}]^{-2}$, we obtain the measure in (5.7). Let $\mathfrak{l}^2_{(1)}$ be the Hilbert space of functions $F(q^{-m})$ on the set $m \in \{0, 1, 2, \cdots\}$ with the scalar product

$$\langle F_1, F_2 \rangle_{(1)} = \sum_{m=0}^{\infty} \frac{(aq;q)_m (-b/a)^m q^{m(m-1)/2}}{(bq, q; q)_m} F_1(q^{-m})\, \overline{F_2(q^{-m})},$$

where the weight function coincides with the measure in (5.7).

Taking into account the modification of the measure and the statement of Theorem 2, we conclude that the q-Meixner polynomials $M_n(q^{-m};\, a, -b/a;\; q)$ and the functions

$$(bq;q)_m(-b)^{-m}q^{-m(m+1)/2} g_n(q^{-m}; a, b|q)$$

constitute an orthogonal basis in the space $\mathfrak{l}^2_{(1)}$.

Proposition 1. *The q-Meixner polynomials $M_n(q^{-m}; a, c; q)$, $n = 0, 1, 2, \cdots$, with the parameters $a = q^{2l-1}$ and $c = -b/a$ do not constitute a complete basis in the Hilbert space $\mathfrak{l}^2_{(1)}$, that is, the q-Meixner polynomials are associated with the indeterminate moment problem and the measure in (5.7) is not an extremal measure for these polynomials.*

Proof. In order to prove this proposition we note that if the q-Meixner polynomials would be associated with the determinate moment problem, then they would constitute a basis in the space of squared integrable functions with respect to the measure from (5.7). However, this is not the case. By the definition of an extremal measure, if the measure in (5.7) would be extremal (see, for example, [12], Chapter VII), then again the set of the q-Meixner polynomials would be a basis in that space. Therefore, the measure is not extremal. Proposition is proved.

Let now $\mathfrak{l}^2_{(2)}$ be the Hilbert space of functions $F(q^{-m})$ on the set $m \in \{0, 1, 2, \cdots\}$, with the scalar product

$$\langle F_1, F_2 \rangle_{(2)} = \sum_{m=0}^{\infty} \frac{(bq;q)_m (-a/b)^m q^{m(m-1)/2}}{(aq, q; q)_m} F_1(q^{-m})\, \overline{F_2(q^{-m})}.$$

The measure above coincides with the orthogonality measure in (5.8) for q-Meixner polynomials $M_n(q^{-m}; b, -a/b;\; q)$, $b < 0$. The following proposition is proved in the same way as Proposition 2.

Proposition 2. *The q-Meixner polynomials $M_n(q^{-m}; b, -a/b; q)$, $n = 0, 1, 2, \cdots$, with $b < 0$ do not constitute a complete basis in the Hilbert space $\mathfrak{l}^2_{(2)}$, that is, these q-Meixner polynomials are associated with the indeterminate moment problem and the measure in (5.8) is not an extremal measure for them.*

According to Propositions 1 and 2, the measures in formulas (5.7) and (5.8), with respect to which the q-Meixner polynomials are orthogonal, are not extremal. As far as we know, explicit forms of extremal measures for the q-Meixner polynomials are not known. It is worth

to mention that extremal measures have been constructed only for the q-Hermite polynomials when $q > 1$ (see [15]).

Note that the complementary set of orthogonal functions to the q-Laguerre polynomials $L_n^{(\alpha)}(x;\, q)$ in the Hilbert space of square integrable functions with respect to the orthogonality measures (which are not extremal) for these polynomials has been constructed in [16].

6. Generating function for big q-Laguerre polynomials

The aim of this section is to derive a generating function for the big q-Laguerre polynomials

$$G(x,t;a,b;\, q) := \sum_{n=0}^{\infty} \frac{(aq,bq;q)_n q^{-n(n-1)/2}}{(q;q)_n} P_n(x;a,b;q)t^n, \tag{6.1}$$

which will be used in the next section. Observe that this formula (6.1) is a bit more general than each of the three instances of generating functions for big q-Laguerre polynomials, given in section 3.11 of [11].

Employing the explicit expression

$$P_n(x;a,b;q) = (b^{-1}q^{-n};q)_n^{-1}\, {}_2\phi_1(q^{-n}, aqx^{-1};\; aq;\; q, x/q)$$

for the big q-Laguerre polynomials, one obtains

$$\begin{aligned} G(x,t;a,b;\, q) &= \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n}(-bqt)^n \sum_{k=0}^{n} \frac{(q^{-n}, aqx^{-1};q)_k}{(aq,q;q)_k} \left(\frac{x}{b}\right)^k \\ &= \sum_{n=0}^{\infty} (aq;q)_n(-bqt)^n \sum_{k=0}^{n} \frac{(-x/b)^k(aqx^{-1};q)_k}{(aq,q;q)_k(q;q)_{n-k}} q^{-nk+k(k-1)/2} \\ &= \sum_{k=0}^{\infty} \frac{(aqx^{-1};q)_k(-x/b)^k}{(aq,q;q)_k} q^{k(k-1)/2} \sum_{m=0}^{\infty} \frac{(aq;q)_{m+k}}{(q;q)_m}(-bqt)^{m+k}q^{-(k+m)k} \\ &= \sum_{k=0}^{\infty} \frac{(aqx^{-1};q)_k}{(q;q)_k}(xt)^k q^{-k(k-1)/2} \sum_{m=0}^{\infty} \frac{(aq^{k+1};q)_m}{(q;q)_m}(-bq^{1-k}t)^m. \end{aligned}$$

By the q-binomial theorem, the last sum equals to $(-abq^2;q)_\infty/(-bq^{1-k}t;q)_\infty$. Since

$$(-bq^{1-k}t;q)_\infty = q^{-k(k-1)/2}(-q/bqt;q)_k(-bqt;q)_\infty,$$

then

$$\frac{(-abq^2;q)_\infty}{(-bq^{1-k}t;q)_\infty} = \frac{(-abq^2;q)_\infty}{(-bqt;q)_\infty} \frac{q^{k(k-1)/2}}{(bt)^k(-1/bt;q)_k}.$$

Thus,

$$\begin{aligned} G(x,t;a,b;\, q) &= \frac{(-abq^2;q)_\infty}{(-bqt;q)_\infty} \sum_{k=0}^{\infty} \frac{(aqx^{-1};q)_k}{(-1/bt,q;q)_k} \left(\frac{x}{b}\right)^k \\ &= \frac{(-abq^2;q)_\infty}{(-bqt;q)_\infty}\, {}_2\phi_1(aqx^{-1},\, 0;\, -1/bt;\; q, x/b). \end{aligned} \tag{6.2}$$

This gives a desired generating function for the big q-Laguerre polynomials.

7. Biorthogonal systems of functions

From the very beginning we could consider an operator

$$A_1 := \alpha q^{J_0/4}\left[(1-bq^{J_0-l})J_+ + q^{J_0-l}J_-\right] q^{J_0/4} - \beta_1 q^{2J_0} + \beta_2 q^{J_0-l}$$

(cf (3.1)), where α, β_1, and β_2 are the same as in (3.1). This operator is well defined, but it is not selfadjoint. Repeating the reasoning of section 3, we find that eigenfunctions of A_1 are of the form

$$\psi_\lambda(x) = \sum_{n=0}^{\infty}(-ab)^{-n/2}q^{-n}\left(\frac{(aq;q)_n}{(q;q)_n}\right)^{1/2} P_n(\lambda;a,b;q)f_n^l(x)$$

$$= \sum_{n=0}^{\infty} a^{-3n/4}(-b)^{-n/2}q^{-n}\frac{(aq;q)_n}{(q;q)_n}P_n(\lambda;a,b;q)x^n, \tag{7.1}$$

where, as before, $a = q^{2l-1}$. The last sum can be summed with the aid of formula (3.11.12) in [11]. We thus have

$$\psi_\lambda(x) = ((-a/b^2)^{1/4}x;q)_\infty \cdot {}_2\phi_1(bq\lambda^{-1},\, 0;\; bq;\;\; q, a^{-3n/4}(-b)^{-1/2}q^{-1}x\lambda).$$

Now we consider another operator

$$A_2 := \alpha q^{J_0/4}(J_+ q^{J_0-l} + J_-(1-bq^{J_0-l}))q^{J_0/4} - \beta_1 q^{2J_0} + \beta_2 q^{J_0-l},$$

where α, β_1, and β_2 are the same as above. This operator is adjoint to the operator $A_1 : A_2^* = A_1$. Repeating the reasoning of section 3, we find that eigenfunctions of A_2 have the form

$$\varphi_\lambda(x) = \sum_{n=0}^{\infty}(-ab)^{-n/2}q^{-n(n+1)/2}\left(\frac{(aq;q)_n(bq;q)_n^2}{(q;q)_n}\right)^{1/2} P_n(\lambda;a,b;q)f_n^l(x)$$

$$= \sum_{n=0}^{\infty} a^{-3n/4}(-b)^{-n/2}q^{-n(n+1)/2}\frac{(aq;q)_n(bq;q)_n}{(q;q)_n}P_n(\lambda;a,b;q)x^n. \tag{7.2}$$

According to the formula (6.2), this function can be written as

$$\varphi_\lambda(x) = \frac{(-abq^2;q)_\infty}{(a^{-3/4}(-b)^{1/2}x;q)_\infty}\,{}_2\phi_1(aq/\lambda, 0;\; a^{3/4}(-b)^{-1/2}q/x;\;\; q, \lambda/b).$$

Let us denote by $\Psi_m(x)$, $m = 0, \pm1, \pm2, \cdots$, the functions

$$\Psi_m(x) = c_m\psi_{aq^{m+1}}(x), \quad m = 0,1,2,\cdots, \qquad \Psi_{-m}(x) = c'_{m-1}\psi_{bq^m}(x), \quad m = 1,2,\cdots, \tag{7.3}$$

and by $\Phi_m(x)$, $m = 0, \pm1, \pm2, \cdots$, the functions

$$\Phi_m(x) = c_m\varphi_{aq^{m+1}}(x), \quad m = 0,1,2,\cdots, \qquad \Phi_{-m}(x) = c'_{m-1}\varphi_{bq^m}(x), \quad m = 1,2,\cdots, \tag{7.4}$$

where c_m and c'_m are given by formulas (4.2) and (4.3).

Writing down the decompositions (7.1) and (7.2) for the functions $\Psi_m(x)$ and $\Phi_m(x)$ (in terms of the orthonormal basis f_n^l, $n = 0, 1, 2, \cdots$, of the Hilbert space $\mathcal{H}_l$) and taking into account the orthogonality relations (5.4)–(5.6) we find that

$$\langle \Psi_m(x), \Phi_n(x)\rangle = \delta_{mn}, \quad m, n = 0, \pm1, \pm2, \cdots.$$

This means that we can formulate the following statement.

Theorem 3. *The set of functions $\Psi_m(x)$, $m = 0, \pm 1, \pm 2, \cdots$, and the set of functions $\Phi_m(x)$, $m = 0, \pm 1, \pm 2, \cdots$, form biorthogonal sets of functions with respect to the scalar product in the Hilbert space $\mathcal{H}_l$.*

8. The classical limit as $q \to 1$

In section 3 we have shown that the operator A, defined by (3.1), is related to the family of big q-Laguerre polynomials (3.2). Namely, the eigenfunctions $\xi_\lambda(x)$ of the operator A can be expanded in terms of the canonical basis functions of the representation T_l^+ of the algebra $U_q(\mathrm{su}_{1,1})$ and coefficients of this expansion are big q-Laguerre polynomials (up to multiplication by a constant factor, see (3.4)).

It is well known that in the limit as $q \to 1$ big q-Laguerre polynomials $P_n(x; a, b; q)$ reduce to classical Laguerre polynomials $L_n^{(\alpha)}(x)$, i.e.,

$$\lim_{q\to 1} P_n(x; q^\alpha, (q-1)^{-1} q^\beta; q) = \frac{L_n^{(\alpha)}(1-x)}{L_n^{(\alpha)}(0)}. \tag{8.1}$$

So, it is natural to expect that there exists a corresponding classical limit of the operator A as $q \to 1$ and $b = q^\beta/(q-1) \to -\infty$. Indeed, this is the case. Bearing in mind that in the case under discussion $a = q^{2l-1}$ and substituting $b = q^\beta/(q-1)$ into (3.1), it is not hard to evaluate that

$$A^{\mathrm{cl}} := \lim_{q\to 1} A = 2(J_1^{\mathrm{cl}} - J_0^{\mathrm{cl}}) + I, \tag{8.2}$$

where I is the identity operator and J_1^{cl} and J_0^{cl} are the generators of the classical Lie algebra $\mathrm{su}(1,1)$, explicitly realized in terms of the first-order differential operators:

$$J_0^{\mathrm{cl}} = x\frac{d}{dx} + l, \qquad J_1^{\mathrm{cl}} = \frac{1}{2}(1+x^2)\frac{x}{dx} + lx. \tag{8.3}$$

Observe that in this particular limit the coefficients of the expansion of the eigenfunctions $\xi_\lambda(x)$ in monomials x^n (see the second line in (3.4)) tend to the Laguerre polynomials $L_n^{(2l-1)}(1-\lambda)$. This means that

$$\xi_\lambda^{\mathrm{cl}}(x) := \lim_{q\to 1}\xi_\lambda(x) = \sum_{n=0}^{\infty} L_n^{(2l-1)}(1-\lambda)x^n = \frac{1}{(1-x)^{2l}} \exp\left(\frac{(\lambda-1)x}{1-x}\right), \tag{8.4}$$

where at the last step we employed the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{xt}{t-1}\right)$$

for the classical Laguerre polynomials.

As a consistency check, it is now easy to evaluate directly with the aid of formulas (8.2)–(8.4) that $A^{\mathrm{cl}}\xi_\lambda^{\mathrm{cl}}(x) = \lambda\xi_\lambda^{\mathrm{cl}}(x)$.

9. Concluding remarks

It is well known that many physical systems admit symmetries with respect to the quantum algebra $U_q(\mathrm{su}_{1,1})$. Then the collection of states of such a system forms a representation space

for this quantum algebra (see, for example, [17]). Hamiltonians of such systems often coincide with representation operators of $U_q(\mathrm{su}_{1,1})$. Since spectra of Hamiltonians are bounded from below, these representations belong, as a rule, to the positive discrete series.

Representation operators of quantum algebras are much more complicated than in the case of Lie algebras. For example, many symmetric representation operators are unbounded and their closures are not selfadjoint operators. Such operators have selfadjoint extensions only if their deficiency indices are equal to each other. In the case of representations of the algebra $U_q(\mathrm{su}_{1,1})$ many representation operators can be equivalently written in the form of a Jacobi matrix. In this case, the selfadjointness of representation operators causes quite definite properties of the corresponding polynomial families. In other words, the study of representation operators leads to deeper understanding of orthogonal polynomials.

In the present paper we have studied in detail those operators in the discrete series representations of $U_q(\mathrm{su}_{1,1})$, which are associated with big q-Laguerre polynomials. In particular, we have found an explicit form of the spectrum for the corresponding selfadjoint representation operator and derived explicitly its eigenfunctions. Using these results, we constructed a system of orthogonal polynomials dual to the big q-Laguerre polynomials. It has occurred that this dual system consists of two sets of q-Meixner polynomials. It is deduced from this fact that the family of q-Meixner polynomials with fixed parameters does not constitute a complete basis in the L^2 space with respect to their orthogonality measure. This means that q-Meixner polynomials correspond to a representation operator with a closure, which is not selfadjoint and has deficiency indices (1,1). Selfadjoint extensions of this operator correspond to so called extremal measures of orthogonality for q-Meixner polynomials. (The knowledge of these measures would give us a possibility to find explicitly spectra of selfadjoint extensions of the operator). Unfortunately, these measures are not known. We hope that the further development of the approach of this paper will enable us to find extremal measures for orthogonal polynomials, which correspond to an indeterminate moment problem.

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